

# Free Vibration Analysis Of Rectangular Plates Using Higher Order Finite Layer Method

**A. A. Abdul-Razzak**

**J. H. Haido**

**Assistant Professor**

**Assistant**

**Lecture**

**Civil Engineering Department,  
Engineering Department,**

**Civil**

**Mosul University, Mosul, Iraq  
University, Dohuk, Iraq**

**Dohuk**

## Abstract

In this paper a higher order finite layer formulation based on the auxiliary nodal surface (ANS) technique for a free vibration analysis of rectangular plates is presented. The free vibration analysis has been performed using the generalized jacobi iteration method, thus finding the natural frequencies and modes of vibration. In this study, authors consider the effects of different types of material and dimensions of the plate on its free vibration analysis.

Many examples have been studied to show the good performance of the higher order finite layer with one ANS for free vibration analysis of plate.

**Keywords:** Finite layer, Free Vibration, Thick Plate.

تحليل الاهتزاز الحر للصفائح المستطيلة باستخدام طريقة الطبقة المحددة ذات النسق العالي

الدكتور أياد امجد عبد الرزاق

السيد جيمس حسدو هيدو

أستاذ مساعد

مدرس مساعد

كلية الهندسة/ جامعة الموصل

كلية الهندسة/ جامعة دهوك

### الخلاصة

في هذا البحث تم استخدام طريقة الطبقة المحددة ذات النسق العالي المعتمدة على الخط العقدي المساعد وذلك لتحليل الاهتزاز الحر لصفائح المستطيلة. تم اجراء تحليل الاهتزاز الحر باستعمال الطريقة التكرارية العامة لجاكوبي وبهذا توجد الترددات الطبيعية وانماط الاهتزاز. الاخذ بنظر الاعتبار تأثير اختلاف المادة والابعاد للصفيحة على تحليل الاهتزاز الحر لها.

تم دراسة عدة امثلة لاطهار الفعالية الجيدة لطريقة الطبقة المحددة ذات النسق العالي دي المساعد في تحليل الاهتزاز الحر للصفيحة.

---

Received 22 Nov. 2005

Accepted 15 Oct. 2006

## Notation:

a	Length of plate
b	Width of plate
[B]	Strain matrix
[C]	Coefficient matrix for displacement function
c	Thickness of plate
[D]	Elasticity matrix
E	Modulus of elasticity
{f}	Displacement function
h	Thickness of layer
m	Particular harmonic number
[M]	Mass matrix
r	Specified number of harmonic terms in x-direction
s	Specified number of harmonic terms in y-direction
[S]	Stiffness matrix
$\left\{ w \right\}$	Vector of nodal displacements
$\left\{ \dot{w} \right\}$	Vector of nodal velocities
$\left\{ \ddot{w} \right\}$	Vector of nodal accelerations
x	Global coordinate in length -wise direction
X <sub>m</sub>	Harmonic function in x-direction
y	Global coordinate in width -wise direction

$Y_m$	Harmonic function in y-direction
$z$	Global coordinate in thickness -wise direction
$\{\phi\}$	Eigen vector
$\rho$	Mass per unit volume
$\nu$	Poisson's ratio
$\omega$	Natural frequency (rad/sec)

### **Introduction:**

Due to scientific and technological development of engineering, rectangular plates have been widely used in different engineering fields. The rectangular plates are commonly used as structural components in aerospace, mechanical, nuclear, marine and structural engineering [1].

A study on the free vibration analysis of plates should be made first before the forced vibration analysis in order to investigate vibrational characteristics of plate (fundamental natural frequency of plate) and to prevent the resonance from occurring. The vibration behavior of structure defined as special spectra consist of an infinite number of frequencies and modes which can be found by knowing geometrical shape, mass distribution, stiffness and boundary conditions of the plate [2].

For years, there has been a large amount of research work on the free vibration analysis of plate structure. In 1970, Sinivas [3] developed a three-dimensional linear, small deformation theory of elasticity solution by the direct method for the free vibration of simply supported thick rectangular plates. The free vibration of isotropic plates with various types of boundary conditions, for three different aspect ratios was investigated by Cheung and Chakrabarti [4] in 1972 by using lower order finite layer technique. The resultant frequencies for the smallest thickness/span ratio are close to that existing for thin plates, while

frequencies for higher thickness/span ratios tend to be of lower value. The closed-form solutions are obtained in 1999 by Meunier and Shenoi [5] for finding the natural frequencies of sandwich plate panels by using higher-order shear deformation theory. Petrolito and Golley [6] in 2000 develop a finite strip-element for the vibration analysis of thick plates. The method uses a combined polynomial and trigonometric interpolation scheme that enables all boundary conditions to be correctly treated. The global equations are derived in the usual manner of the finite element method, and natural frequencies of vibration can be found by solving a linear eigenvalue problem.

In 2003, Sheikh et al. [7] analyzed the free vibration for both the thin and thick plates. The solution depends on Reissner-Mindlin theory by adopting a new type of triangular element with three nodes at corners, three others at mid of the three sides and one internal node at the center of element.

In the present study, a higher order finite layer with a second order polynomial has been used for the free vibration analysis of plates. The higher order layer produced by introducing an auxiliary nodal surface at mid the distance between the upper and lower surfaces of the lower order layer.

### **Finite Layer Method:**

The plate was divided into a number of horizontal layers in the direction of the thickness of a layer. These layers may be of lower or higher order and for finite layer formulation the layers may be imagined as nodes as shown in (Fig. 1). By selecting functions satisfying the boundary conditions in two directions, the philosophy of the finite strip method can be extended to layered systems. The resulting method is called the finite layer method (FLM), and this method is useful for layered materials, rectangular in plan form [8].

The general form of the displacement function of a layer element is given as a product of polynomials in the thickness direction and continuously differentiable smooth series in the other two directions.

With the stipulation that such series (called basic functions) should satisfy the boundary conditions at the edges of the layer thus a three-dimensional problem is reduced to one-Dimensional problem.

For a layer element the generalized displacement functions can be constructed as [9, 10]:

$$u = \sum_{m=1}^r \sum_{n=1}^s u_{mn}(z) \cdot X'_m(x) \cdot Y_n(y)$$

$$v = \sum_{m=1}^r \sum_{n=1}^s v_{mn}(z) \cdot X_m(x) \cdot Y'_n(y)$$

$$w = \sum_{m=1}^r \sum_{n=1}^s w_{mn}(z) \cdot X_m(x) \cdot Y_n(y)$$

.....(1)

Where  $X_m(x)$ ,  $X'_m(x)$ ,  $Y_n(y)$  and  $Y'_n(y)$  are the basic functions and their first derivative.  $u_{mn}(z)$ ,  $v_{mn}(z)$  and  $w_{mn}(z)$  are the interpolating functions. The  $X_m(x)$  and  $Y_n(y)$  are used in the expression for  $u$  and  $v$  respectively, because the relationships  $u = A(\partial w / \partial x)$  and  $v = B(\partial w / \partial y)$  in the linear plate theory are:-

$$u_{mn}(z) = [C] \cdot \{U\}_{mn}$$

$$v_{mn}(z) = [C] \cdot \{V\}_{mn}$$

$$w_{mn}(z) = [C] \cdot \{W\}_{mn}$$

.....(2)

Where  $[C]$  represent the shape functions in the z-direction with the higher order finite layer (corresponding to the layer with an auxiliary nodal surface):

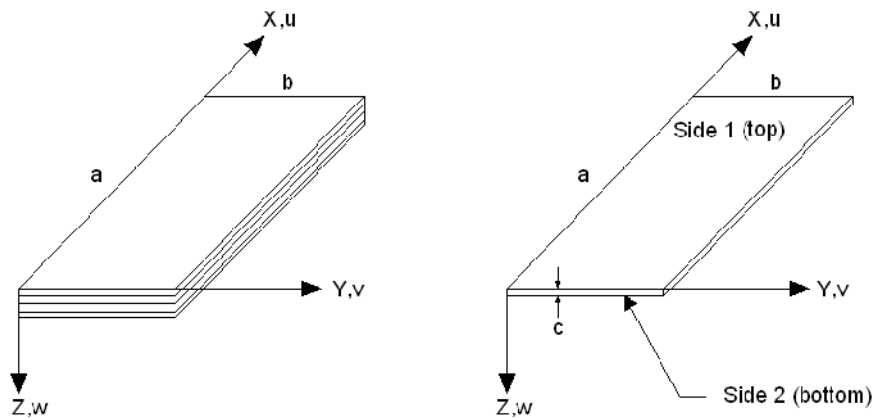
$$[C] = \left[ 1 + \frac{2z^2}{h^2} - \frac{3z}{h}, -\frac{4z^2}{h^2} + \frac{4z}{h}, \frac{2z^2}{h^2} - \frac{z}{h} \right] \dots\dots\dots(3)$$

$$\{U\}_{mn} = [U_{mni}, U_{mnj}, U_{mnk}]^T,$$

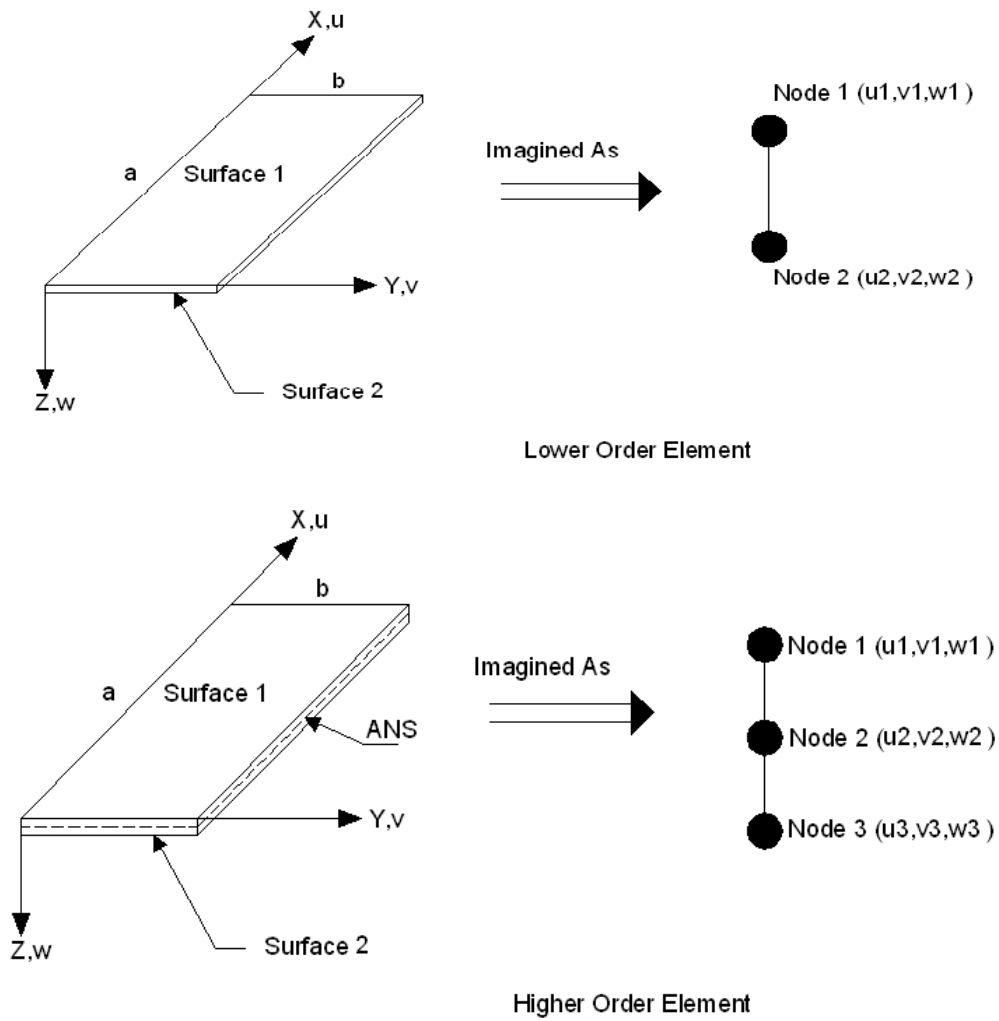
$$\{V\}_{mn} = [V_{mni}, V_{mnj}, V_{mnk}]^T, \quad \{W\}_{mn} = [W_{mni}, W_{mnj}, W_{mnk}]^T \dots\dots\dots(4)$$

Where the unknown parameters:

$U_{mni}, U_{mnk}, U_{mnj}, V_{mni}, V_{mnk}, V_{mnj}, W_{mni}, W_{mnk}$  and  $W_{mnj}$  represent the amplitudes of the corresponding quantities at the surfaces i,k and j respectively.



**Plate Idealization**



**(Fig. 1) Finite Layer Method**

The simply supported layer is by far the most important and useful for such a layer element the generalized displacement functions in equations (1) can be taken as:

$$u = \sum_{m=1}^r \sum_{n=1}^s [C] \cdot \{U\}_{mn} \cdot \cos \frac{m\pi}{a} x \cdot \sin \frac{n\pi}{b} y$$

$$v = \sum_{m=1}^r \sum_{n=1}^s [C] \cdot \{V\}_{mn} \cdot \sin \frac{m\pi}{a} x \cdot \cos \frac{n\pi}{b} y$$



$$w = \sum_{m=1}^r \sum_{n=1}^s [C] \cdot \{W\}_{mn} \cdot \sin \frac{m\pi}{a} x \cdot \sin \frac{n\pi}{b} y$$

.....(5)

In which:

a = Length of layer

b = Width of layer

r, s = Number of harmonic in x, y direction respectively.

The generalized displacement functions (1) for a layer can be written in a concise notation as:

$$\{f\} = \sum_{m=1}^r \sum_{n=1}^s [N] \cdot \{W\}_{mn} = [N] \{W\}$$

.....(6)

Where [N] combines together the series and the shape functions:

$$\{W\}_{mn} = \left\{ \begin{array}{l} \{U\}_{mn} \\ \{V\}_{mn} \\ \{W\}_{mn} \end{array} \right\}$$

.....(7)

The strain-displacement relationship is:

$$\{\varepsilon\} = \sum_{m=1}^r \sum_{n=1}^t [B]_{mn} \cdot \{W\}_{mn}$$

.....(8)

Where [B] matrix is referred to as the generalized strain matrix.

The stress-strain relationship is:

$$\{\sigma\} = [D] \sum_{m=1}^r \sum_{n=1}^t [B]_{mn} \cdot \{W\}_{mn}$$

.....(9)

Where [D] is the elasticity matrix of the material for the particular layer under consideration which can be isotropic or orthotropic.

**Derivation of the Stiffness Matrix:**

The standard formula for the stiffness matrix is [9, 10]:

$$[S]_{mnpq} = \int_0^a \int_0^b \int_0^c [B]_{mn}^T [D] [B]_{pq} dx.dy.dz$$

.....(10)

Where [S]<sub>mnpq</sub> is the generalized stiffness matrix which has the following expanded form:

$$[S] = \begin{bmatrix} S_{1111} & S_{1112} & \dots & S_{111s} & S_{1121} & \dots & S_{112s} & \dots & S_{11rs} \\ S_{1211} & S_{1212} & \dots & S_{121s} & S_{1221} & \dots & S_{122s} & \dots & S_{12rs} \\ \dots & & & & & & & & \\ S_{1s11} & S_{1s12} & & S_{1s1s} & S_{1s21} & & S_{1s2s} & & S_{1srs} \\ S_{2111} & S_{2112} & & S_{211s} & S_{2121} & & S_{212s} & & S_{21rs} \\ \dots & & & & & & & & \\ S_{2s11} & S_{2s12} & & S_{2s1s} & S_{2s21} & & S_{2s2s} & & S_{2srs} \\ \dots & & & & & & & & \\ S_{rs11} & S_{rs12} & & S_{rs1s} & S_{rs21} & & S_{rs2s} & & S_{rsrs} \end{bmatrix}$$

.....(11)

For a simply supported case owing to the orthogonality of trigonometric series, the terms of the series can be decoupled and may be solved separately.

### Derivation of the Consistent Mass Matrix:

As it has been mentioned, the displacement function of any layer has the form as in equation (6):

$$[N]_{mpq} = [C]X_m \cdot Y_n$$

.....(12)

Thus the consistent mass matrix for a layer is developed as:

$$[M]_{mn} = \int_0^a \int_0^b \int_0^c \rho \cdot [N]_{mn}^T \cdot [N]_{pq} dx \cdot dy \cdot dz$$

.....(13)

Where  $\rho$  is the mass per unit volume.

For a higher order layer element  $[N]_{mn}$  can be written as follows:

$$[M]_{mn} = \begin{bmatrix} c1XY_{x_n} & 0 & 0 & c2XY_{x_n} & 0 & 0 & c3XY_{x_n} & 0 & 0 \\ 0 & c1XY_{x_n} & 0 & 0 & c2XY_{x_n} & 0 & 0 & c3XY_{x_n} & 0 \\ 0 & 0 & c1XY_{x_n} & 0 & 0 & c2XY_{x_n} & 0 & 0 & c3XY_{x_n} \end{bmatrix}$$

.....(14)

Where

$$c1 = 1 + \frac{2z^2}{h^2} - \frac{3z}{h}$$

$$c2 = -\frac{4z^2}{h^2} + \frac{4z}{h}$$

$$c3 = \frac{2z^2}{h^2} - \frac{z}{h}$$

.....(15)

### Free Vibration Analysis:

The general matrix differential equation of the system under dynamic equilibrium is:

$$\dots\dots\dots(16)$$

$$[M]\{\ddot{W}\} + [A]\{\dot{W}\} + [S]\{W\} = \{F\}$$

Where  $\{W\}$  represents the nodal global degree of freedom.  $[M]$ ,  $[A]$  and  $[S]$  are the mass, damping and stiffness matrix of the structure, respectively;

$\{\ddot{W}\}$  and  $\{\dot{W}\}$  are the acceleration and velocity amplitude vectors respectively. The load factor  $\{F\}$  represents the nodal loads.

By making  $\{F\}$  zero in equation (16), the equation of free vibration of the undamped plates is:

$$\dots\dots\dots(17)$$

$$[M]\{\ddot{W}\} + [S]\{W\} = 0$$

The standard eigen value formulation of problem is therefore:

$$\dots\dots\dots(18)$$

$$([S]_{mn} - \omega_{mn}^2 [M]_{mn})\{W\} = \{0\}$$

Where  $\omega_{mn}^2$  is an eigenvalue and  $\{W\}$  is the corresponding eigenvector. The generalized Jacobi Iteration is employed for solving equation (18) to determine the natural frequencies and their corresponding mode shapes.

Many computer programs in FORTRAN language have been developed to solve numerical problems in free vibration analysis for rectangular plates. The original programs are those provided in 1991 by Majeed [10]

who used it in static analysis for rectangular plates. These programs are suitable for dynamic analysis of rectangular plates which have different boundary conditions.

### **Numerical Examples:**

In order to demonstrate the capability and efficiency of the formulation presented and the reliability of the higher order finite layer with one auxiliary nodal surface in dealing with free vibration analysis of plates, typical examples have been studied.

#### **Example-1: Free vibration of a thin isotropic plate simply-supported at all sides:**

A thin plate simply-supported at the four edges is analyzed by the higher order finite layer taking different aspect ratio ( $a / b$ ). Natural frequencies for the lowest six modes of vibration obtained and compared with the results reported by Leissa [11] and presented in Table-1. The properties of the plate are:  $a= 1$  m,  $b=$  variable value,  $\nu = 0.1660$ ,  $E= 1.17 \times 10^7$  kN/m<sup>2</sup>,  $\rho = 100$  kN/m<sup>3</sup>,  $c= 0.01$  m.

**Table (1) Natural frequencies of simply supported thin isotropic plate**

<b>Mode No.</b>	<b>Method</b>	<b>Aspect ratio (a/b)</b>				
		<b>0.4</b>	<b>2/3</b>	<b>1</b>	<b>1.5</b>	<b>2.5</b>
<b>1</b>	<b>Leissa</b>	<b>11.4487</b>	<b>14.2561</b>	<b>19.7392</b>	<b>32.0762</b>	<b>71.5564</b>
	<b>Present Study</b>	<b>11.4500</b>	<b>14.2500</b>	<b>19.7340</b>	<b>31.8400</b>	<b>71.4000</b>
<b>2</b>	<b>Leissa</b>	<b>16.1862</b>	<b>27.4156</b>	<b>49.3480</b>	<b>61.6850</b>	<b>101.1634</b>

	<b>Present Study</b>	<b>16.1700</b>	<b>27.3800</b>	<b>49.2640</b>	<b>61.3500</b>	<b>100.92</b>
<b>3</b>	<b>Leissa</b>	<b>24.0818</b>	<b>43.8649</b>	<b>49.3480</b>	<b>98.6960</b>	<b>150.5115</b>
	<b>Present Study</b>	<b>24.0500</b>	<b>43.80</b>	<b>49.2640</b>	<b>97.60</b>	<b>150.048</b>
<b>4</b>	<b>Leissa</b>	<b>35.1358</b>	<b>49.3480</b>	<b>78.9568</b>	<b>111.0330</b>	<b>219.5987</b>
	<b>Present Study</b>	<b>35.0830</b>	<b>49.2640</b>	<b>78.7900</b>	<b>110.5300</b>	<b>218.70</b>
<b>5</b>	<b>Leissa</b>	<b>41.7950</b>	<b>57.0244</b>	<b>98.6960</b>	<b>128.3049</b>	<b>256.6097</b>
	<b>Present Study</b>	<b>40.9922</b>	<b>56.9200</b>	<b>98.4600</b>	<b>127.1070</b>	<b>255.4300</b>

**Example-2: Free vibration of a square isotropic plate having different boundary conditions:**

In this example the frequency parameters ( $\varphi = \omega a \sqrt{2\rho(1+\nu)/E}$ ) are to be found for a thick square plate with different boundary conditions, the material and geometrical properties of the plate are as following:

Length (a) = Width (b) = 250mm

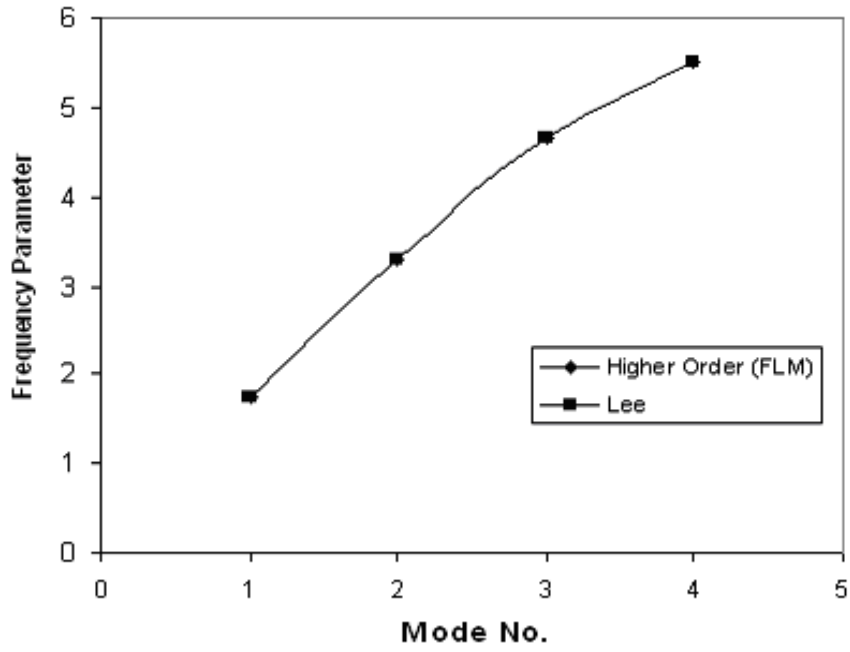
Thickness (h) = 25mm

Density ( $\rho$ ) = 7850kg/m<sup>3</sup>

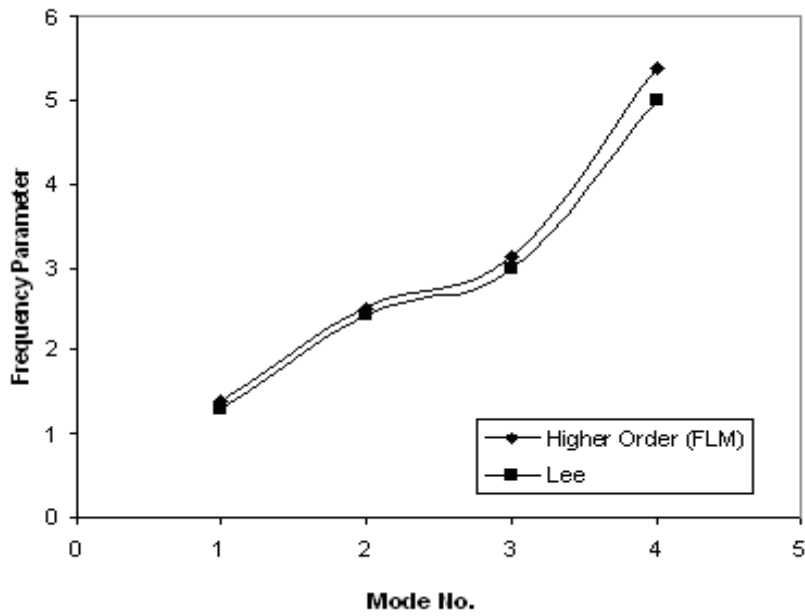
Modulus of Elasticity (E) = 2x10<sup>5</sup>MPa

Poisson Ratio ( $\nu$ ) = 0.3

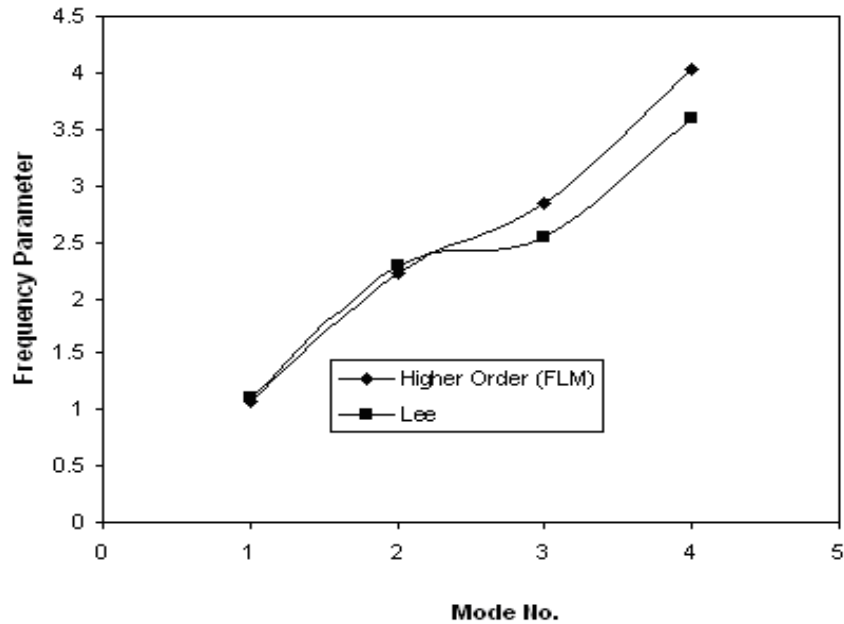
(Fig. 2), (Fig. 3) and (Fig. 4) show the comparison between the values resulted from present method and those found by Lee [12].



**(Fig. 2) Frequencies for the plate with fixed supports**



**(Fig. 3) Frequencies for the plate with fixed-simply-fixed- simply supports**



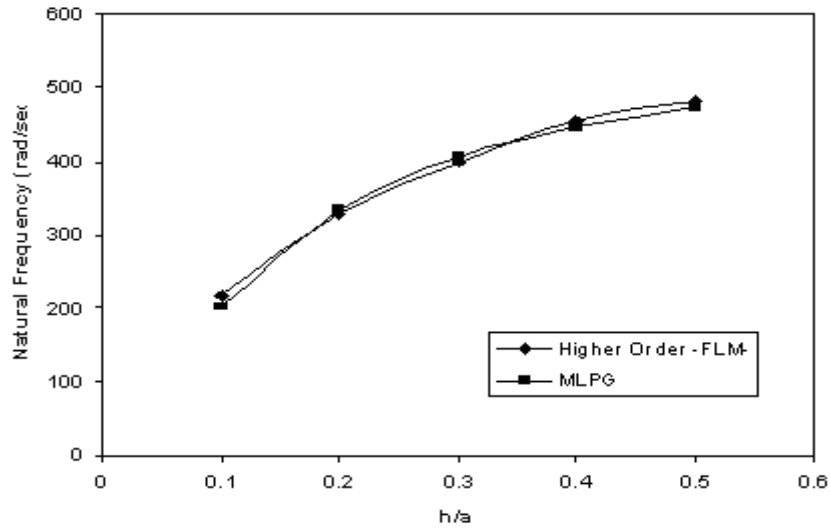
**(Fig. 4) Frequencies for the plate with simply-fixed-simply- simply supports**

**Example-3: Effect of the thickness of plate on frequency:**

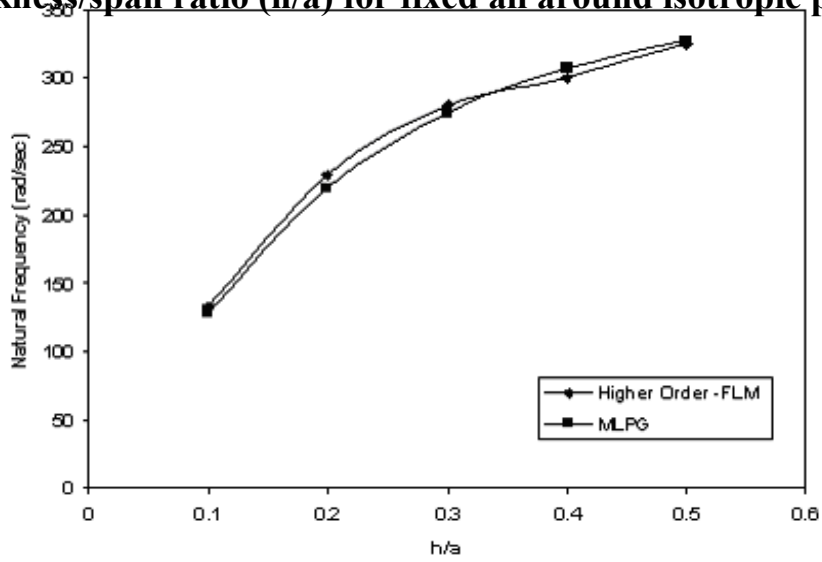
Results are presented in terms of the natural frequency (rad/sec). The plate has the same material and geometrical properties in the previous example except the thickness will be variable here.

In Figs. (5) and (6) the relation between the lowest natural frequency and the thickness/span ratio ( $h/a$ ) are illustrated for two plates having different boundary conditions, the results are compared with the available data computed for the same plates by Ref. (13).





**(Fig. 5) Relation between the lowest natural frequency and thickness/span ratio ( $h/a$ ) for fixed all around isotropic plate**



**(Fig. 6) Relation between the lowest natural frequency and thickness/span ratio (h/a) for (fixed-simply-fixed- simply supports) isotropic plate**

**Example-4: Orthotropic plate:**

In order to compute natural frequencies of a simply supported orthotropic square plate. Elastic constants as that given by Srinivas et al 1970 [3] are taken.

$$D = \begin{bmatrix} 160 & 37.3 & 1.72 & 0 & 0 & 0 \\ & 86.87 & 15.72 & 0 & 0 & 0 \\ & & 84.81 & 0 & 0 & 0 \\ & & & 25.58 & 0 & 0 \\ \text{Sym} & & & & 42.68 & 0 \\ & & & & & 42.06 \end{bmatrix} \text{ GPa}$$

Table (2) lists frequencies parameter of different modes obtained by various methods.

**Table (2) Frequency parameters of simply supported moderately thick orthotropic plate with h/a=0.1**

<b>Method Mode No.</b>	<b>Present Study</b>	<b>Qian [13]</b>	<b>Kant [14]</b>	<b>Sirivas [3]</b>
<b>1</b>	<b>0.045</b>	<b>0.0477</b>	<b>0.0474</b>	<b>0.0474</b>
<b>2</b>	<b>0.100</b>	<b>0.1028</b>	<b>0.1033</b>	<b>0.1033</b>
<b>3</b>	<b>0.120</b>	<b>0.1236</b>	<b>0.1188</b>	<b>0.1188</b>
<b>4</b>	<b>0.170</b>	<b>0.1726</b>	<b>0.1694</b>	<b>0.1694</b>
<b>5</b>	<b>0.187</b>	<b>0.1850</b>	<b>0.1888</b>	<b>0.1888</b>

<b>6</b>	<b>0.234</b>	<b>0.2172</b>	<b>0.2181</b>	<b>0.2180</b>
<b>7</b>	<b>0.245</b>	<b>0.2439</b>	<b>0.2476</b>	<b>0.2475</b>
<b>8</b>	<b>0.273</b>	<b>0.2727</b>	<b>0.2625</b>	<b>0.2624</b>
<b>9</b>	<b>0.300</b>	<b>0.3066</b>	<b>0.2969</b>	<b>0.2969</b>
<b>10</b>	<b>0.372</b>	<b>0.3307</b>	<b>0.3319</b>	<b>0.3319</b>

### **Conclusions:**

There were many advantages from using finite layer method specially for analyzing the plates which compound from different layers. The higher order finite layer method is used successfully to study the vibration characteristics by finding the natural frequencies of a rectangular plate subjected to different edge conditions. Computed frequencies were found to agree very well with the corresponding results available in the literature.

It is clear from the results that the natural frequency of the plate will be increased by increasing the aspect ratio ( $a/b$ ) and the plate thickness.

### **References:**

1. Zhou D., Cheung Y.K., Au F.T.K., and Lo S.H. "Three-dimensional vibration analysis of thick plates using Chebyshev polynomial and Ritz method", *International Journal of Solid and Structure*. Vol. 39, July 2002, pp. 6339-6353.
2. Chen H.L., Spyrakos C.C., and Venkatesh G. "Evaluating structural deterioration by dynamic response", *Journal of Structural Engineering*. Vol. 121, No.8, Aug. 1995, pp. 1197-1204.
3. Srinivas S., Jaga Rao C.V., and Rao A.K. "An exact analysis for vibration of simply-supported homogeneous and laminated thick

rectangular plates", *Journal of Sound and Vibration. Vol. 12, No.2, 1970, pp. 187-199.*

4. Cheung Y.K., and Chakrabarti S. "Free vibration of thick layered rectangular plates by a finite layer method", *Journal of Sound and Vibration. Vol. 21, No. 3, 1972, pp. 277-284.*

5. Meunier M., and Sheno R. A. "Free vibration analysis of composite sandwich plates", *Journal of Sound and Vibration. Vol. 213, 1999, pp. 715-727.*

6. Petrolito J., and Golley B.W. "Vibration of thick plates using finite strip-elements", *ANZIAM Journal. Vol. 42, Aug. 2000, pp. 1137-1155.*

7. Sheikh A.H., Dey P. and Sengupta D. "Vibration of thick and thin plates using a new triangular element", *Journal of Engineering Mechanics. Vol. 129, No. 11, Nov. 2003, pp. 1235-1244.*

8. Cheung Y.K., Lo S.H., and Leung A.Y.T. (1996) Finite Element Implementation. *Blackwell Science.*

9. Cheung Y.K. (1976) Finite Strip Method in Structural Analysis, *Pergamon Press.*

10. Majeed A.S. "Analysis of thick rectangular plates by the finite layer method". *M.Sc. Thesis. Dept. of Civil Eng., University of Mosul, Mosul 1991.*

11. Leissa A.W. "The free vibration of rectangular plates", *Journal of Sound and Vibration. Vol. 31, No. 3, 1973, pp. 257-293.*

12. Lee S.J. "Free vibration analysis of plates by using a four-node finite element formulated with assumed natural transverse shear strain", *Journal of Sound and Vibration. Vol. 278, 2004, pp. 657-684.*

13. Qian L.F., Batra R.C. and Chen L.M. "Free and forced vibrations of thick rectangular plates using higher order shear and normal deformable plate theory and meshless Petrov-Galerkin (MLPG) method", *CMES. Vol. 4, No. 5, 2003, pp. 519-534.*

14. Kant T. and multiplayer plates based on higher order refined theories", *Journal Swaminathan K. "Free vibration of isotropic, orthotropic and of Sound and Vibration. Vol. 241, No. 2, 2001, pp. 319-327.*