

In Plates And Multi-Material Beams

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Abstract

The behaviors of structural waves (axial, shear and bending) are studied in multi-material beams and single material plates using the method of characteristics (MOC). The characteristics equations in one dimensional problems are extended and derived to cover the two dimensional problems and to make the method be applicable to study the wave behavior in plates.

The results showed that the structural waves changed their forms during their propagation in the multi-material beams and some reduction is occurred in the natural frequency of the beam. The propagation of structural waves in plates showed same behavior as in beams, so the propagation in two directions can be separated and each direction can be considered individually as one dimensional problem to simplify the characteristics equations and saving the computing time and solution techniques.

KEYWORDS: Beams, Method of Characteristics, Plates, Structural waves and Vibration.

طريقة الخواص (MOC) في البلاطات والعتبات المتعددة المواد

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الكلية التقنية – أربيل

الخلاصة

تمت دراسة خواص الامواج الانشائية المحورية، القص والانثناء (axial, shear and bending) في البلاطات المكونة من مادة واحدة والعتبات المكونة من أكثر من مادة واحدة باستخدام طريقة الخواص (Method of Characteristics)، حيث تم اشتقاق وتطوير معادلات الخواص للمسائل ذات البعد الواحد (Beams) وكذلك تطبيق هذه الطريقة على المسائل ذات البعدين (Plates).

وأظهرت النتائج بأن الامواج الانشائية (المحورية، القص والانثناء) تغير من أشكالها خلال تقدمها في العتبات المكونة من أكثر من مادة واحدة و يحصل بعض النقص في قيمة التردد الطبيعي للعتبة وأظهرت أيضا أن تقدم الامواج في البلاطات مشابه للعتبات في كل اتجاه وبذلك يمكن معاملة المسائل ذات البعدين كأتجاهين منفصلين كما في العتبات لتبسيط معادلات الخواص في البلاطات وأختصار زمن الحساب وتسهيل عملية التحليل.

Notations:

A = cross section area of the beam.

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C = speed of the axial wave.

Cs = speed of the shear wave.

E = modulus of elasticity.

F , Q , M = axial force, shear force and bending moment at node (i) and time (t1) in beams.

F_X , F_Y = in-plane forces in x and y directions respectively at node (i) and time (t1) in plates.

G = shear modulus.

h = plate thickness.

I = moment of inertia.

ks = shear coefficient.

M_X , M_Y = bending moments about x and y axis respectively at node (i) and time (t1) in plates.

M_{XY} , M_{YX} = twisting moments at node (i) and time (t1) in plates.

Q_X , Q_Y = shear forces in xz and yz planes respectively at node (i) and time (t1) in plates.

Q'_L , V'_L and ψ'_L = shear force, shear velocity and angular velocities along the line (PL').

Q'_R , V'_R and ψ'_R = shear force, shear velocity and angular velocities along the line (PR').

U , V , ψ = axial, shear and angular velocities at node (i) and time (t1) in beams.

U = axial velocity = du/dt ; u = axial displacement.

V = transverse velocity = dv/dt ; v = transverse displacement.

ψ = angular velocity = $d\psi_O/dt$; ψ_O = angular displacement.

Δ_O is the deflection at node (i) and time (t1).

Δ_L is the deflection at node (i-1) and time (t1).

Δ_R is the deflection at node (i+1) and time (t1).

$\Delta x, \Delta y$ = the finite difference intervals in x and y directions respectively.

ρ = density of the material.

ν = Poisson's ratio.

Introduction:

The method of characteristics (MOC) is a widely used explicit method of solution for hyperbolic differential equations [1-7]. It is frequently used to model a variety of wave propagation phenomena. Closed conduit, open channel and ground water flows have been analyzed using this technique. Many workers have used the technique to simulate wave propagation in different media. The method expanded due to three reasons:

- 1- The possibility in reducing the partial differential equations to straight forward ordinary differential equations.
- 2- It's mathematics in time – space coordinates, so it is suitable to use for wave propagation problems.
- 3- The computer resources required are relatively small.

Because of the complexity of the wave form, some assumptions are made to simplify the theoretical models. The most important assumptions in the case of axial stress waves is usually that plane section remain plane, little is known about the validity of this assumption during wave propagation. The axial wave in elastic and prismatic members will propagate in unchanged form while flexural waves do not propagate in an unchanging form, because flexural waves consist of a combination of bending and shear waves which propagate simultaneously, but at different wave speeds.

The phenomena of wave propagation and the (MOC) are described by different workers. Vardy and Al-Sarraj [1], applied the (MOC) to simulate a coupled axial, flexural and torsional vibrations in single structural members supporting discrete masses, they showed that this method has useful advantages over the conventional analysis.

Plass [2] applied the (MOC) to solve the vibration of Timoshenko beam for various types of support conditions and the results showed good agreement with experimental data. Chou and Mortimer [3], applied the (MOC) to study the elastic wave problem in different types of structures (Timoshenko beam, plates, bars and sheets), and

Atkins and Hunter [4] considered the propagation of longitudinal elastic waves around right angle corners in rods of square cross section, they showed that the theoretical analysis based on the assumption that the filament approximation is valid every where except in the vicinity of the corner, the results found good agreement with the experimental observations.

Al Mousawi [5], considered the effect of discontinuous in cross section on the behavior of flexural wave in beams, the results showed good agreement with the experimental works. Al-Sarraj and Al-Daami [6], investigated the transient dynamic response of non prismatic structural members, using the (MOC), they showed that the method is capable to predict the natural frequency of vibration, the results showed good agreement with that obtained by finite element method. Al-Sarraj et al. [7], include non linear characteristics in the analysis of axially loaded bar, using (MOC) and they showed that this method is suitable to simulate the transient and dynamic behavior of axially loaded bar beyond the elastic stage.

Peng [8 and 9], introduced an explicit acoustical wave propagator method to investigate the flexural wave propagation and dynamic stress concentration in a multi-stepped plate and one dimensional structures

with discontinuities. A new combined scheme with Chebyshev polynomial expansion and fast Fourier transformation is used and compared with Euler and exact analytical solution. Luangvilai et al. [10], examined the propagation of guided lamb waves in fiber-reinforced polymer bounded components, establishing the effectiveness of combining laser ultrasonic techniques with a time frequency representation to experimentally measure the dispersion curves of a concrete components repaired with a fiber-reinforced polymer plate. Also Komatitsch et al. [11] introduced a numerical approach for modeling elastic wave propagation in two and three dimensional fully anisotropic media based upon a spectral element method.

In the present study, the behaviors of structural waves (axial, shear and bending) are studied in multi-material beams and single material plates using the method of characteristics (MOC). The characteristics equations in one dimensional problems are extended and derived to cover the two dimensional problems and the method be applicable to study the wave behavior in plates.

Formulation:

The propagation of uncoupled axial and flexural waves along one dimensional prismatic elastic member has been discussed by Timoshenko [12]. It is assumed that plane section remain plane in the case of axial wave, but in case of flexural waves, the presence of shear waves cause section to distort. The derivations of the characteristics equations are given in [13], the final finite difference form of characteristics equation are shown below:

1- Uncoupled axial – flexural waves:

Axial waves:

$$\text{Along } C=dx/dt \quad ; (F_P - F_L) - \rho AC (U_P - U_L) = 0 \quad \text{---}$$

------(1)

$$\text{Along } C= - dx/dt; (F_P - F_R) + \rho AC (U_P - U_R) = 0 \quad \text{---}$$

------(2)

Flexural waves – Lateral:

$$\text{Along } C_s=dx/dt \quad ; (Q_P - Q'_L) - \rho AC_s (V_P - V'_L) = - \rho AC_s \Delta x/2 (\psi_P + \psi'_L) \quad \text{-----(3)}$$

$$\text{Along } C_s= - dx/dt; (Q_P - Q'_R) + \rho AC_s (V_P - V'_R) = - \rho AC_s \Delta x/2 (\psi_P + \psi'_R) \quad \text{---(4)}$$

Flexural waves – Angular:

$$\text{Along } C=dx/dt \quad ; (M_P - M_L) - \rho IC (\psi_P - \psi_L) = - \Delta x/2 (Q_P + Q_L) \quad \text{-----(5)}$$

$$\text{Along } C= - dx/dt; (M_P - M_R) + \rho IC (\psi_P - \psi_R) = \Delta x/2 (Q_P + Q_R) \quad \text{-----(6)}$$

Where:

$C = \text{speed of the axial wave} = \sqrt{E/\rho}$.

$C_s = \text{speed of the shear wave} = \sqrt{ks G/\rho}$.

$\rho = \text{density of the material}$.

$E = \text{modulus of elasticity}$.

$G = \text{shear modulus}$.

A = cross section area of the beam.

I = Moment of inertia.

F, Q, M = axial force, shear force and bending moment at node (i) and time (t1) in beams.

U, V, ψ = axial, shear and angular velocities at node (i) and time (t1) in beams.

U = axial velocity = du/dt ; u = axial displacement.

V = transverse velocity = dv/dt ; v = transverse displacement.

ψ = angular velocity = $d\psi_O/dt$; ψ_O = angular displacement.

F_P, Q_P, M_P = axial force, shear force and bending moment at node (i) and time (t2) in beams.

U_P, V_P, ψ_P = axial, shear and angular velocities at node (i) and time (t2) in beams.

F_L, Q_L, M_L = axial force, shear force and bending moment at node (i-1) and time (t1) in beams.

U_L, V_L, ψ_L = axial, shear and angular velocities at node (i-1) and time (t1) in beams.

F_R, Q_R, M_R = axial force, shear force and bending moment at node (i+1) and time (t1) in beams.

U_R, V_R, ψ_R = axial, shear angular velocities at node (i+1) and time (t1) in beams.

ks = shear coefficient depends on the cross section shape [14].

Q'_L, V'_L and ψ'_L = shear force, shear velocity and angular velocities along the line (PL') as shown in Fig.(1).

Q'_R , V'_R and ψ'_R = shear force, shear velocity and angular velocities along the line (PR') as shown in Fig.(1).

2- Coupled axial – flexural waves: in case of coupled axial and flexural waves, the MOC equations in finite difference form are shown below [13]

Axial waves:

$$\text{Along } C=dx/dt \quad ; (F_P - F_L) - \rho AC (U_P - U_L) = \rho I (\psi_P \psi_O - \psi_L^2) \quad -$$

------(7)

$$\text{Along } C= - dx/dt; (F_P - F_R) + \rho AC (U_P - U_R) = \rho I (\psi_P \psi_O - \psi_R^2) \quad -$$

------(8)

Flexural waves – Lateral:

$$\text{Along } C_s=dx/dt \quad ; (Q_P - Q'_L) - \rho AC_s (V_P - V'_L) = - \rho AC_s \Delta x/2 (\psi_P + \psi'_L) \quad \text{-----}(9)$$

$$\text{Along } C_s= - dx/dt; (Q_P - Q'_R) + \rho AC_s (V_P - V'_R) = - \rho AC_s \Delta x/2 (\psi_P + \psi'_R) \quad \text{---}(10)$$

Flexural waves – Angular:

$$\text{Along } C=dx/dt; (M_P - M_L) - \rho IC (\psi_P - \psi_L) = - \Delta x/2 (Q_P + Q_L) + (F_P + F_L)(\Delta_O - \Delta_L)/2 \quad \text{-----}(11)$$

$$\text{Along } C= - dx/dt; (M_P - M_R) + \rho IC (\psi_P - \psi_R) = \Delta x/2 (Q_P + Q_R) + (F_P + F_R)(\Delta_O - \Delta_R)/2 \quad \text{-----}(12)$$

stress – strain relationships in (x and y) directions and plate equations [14 and 16]. The final finite difference form can be written as below:

Axial waves – x-direction:

$$\text{Along } C=dx/dt \quad ; (F_{XP} - F_{XL}) - K1 (U_P - U_L) - K2 (V_P - V_L) = 0$$

-----(13)

$$\text{Along } C= - dx/dt; (F_{XP} - F_{XR}) + K1 (U_P - U_R) + K2 (V_P - V_R) = 0$$

-----(14)

Axial waves – y-direction:

$$\text{Along } C=dy/dt \quad ; (F_{YP} - F_{YL}) - K2 (U_P - U_L) - K1 (V_P - V_L) = 0$$

-----(15)

$$\text{Along } C= - dy/dt; (F_{YP} - F_{YR}) + K2 (U_P - U_R) + K1 (V_P - V_R) = 0$$

-----(16)

In which: $K1 = \rho h C / (1 - \nu^2)$, $K2 = \nu K1$, (ν) is the Poisson's ratio and (h) is the plate thickness. And assuming the wave speed in x and y directions are equal.

Flexural waves – Lateral x- direction:

$$\text{Along } C_s=dx/dt \quad ; (Q_{XP} - Q'_{XL}) - \rho h C_s (V_P - V'_L) = - \rho h C_s \Delta y / 2 (\psi_{YP} + \psi'_{YL})$$

-----(17)

$$\text{Along } C_s = -dx/dt; (Q_{XP} - Q'_{XR}) + \rho h C_s (V_P - V'_R) = -\rho h C_s \Delta y/2 (\psi_{YP} + \psi'_{YR}) \quad \text{-----(18)}$$

Flexural waves – Lateral y- direction:

$$\text{Along } C_s = dy/dt; (Q_{YP} - Q'_{YL}) - \rho h C_s (U_P - U'_L) = -\rho h C_s \Delta x/2 (\psi_{XP} + \psi'_{XL}) \quad \text{-----(19)}$$

$$\text{Along } C_s = -dy/dt; (Q_{YP} - Q'_{YR}) + \rho h C_s (U_P - U'_R) = -\rho h C_s \Delta x/2 (\psi_{XP} + \psi'_{XR}) \quad \text{-----(20)}$$

Flexural waves – Angular x- direction:

$$\text{Along } C = dx/dt; (M_{XP} - M_{XL}) - (M_{YXP} - M_{YXL}) = -\Delta x/2(Q_{XP} + Q_{XL}) + \Delta y/2 \rho I C (\psi_{XP} - 2\psi_{XL} + \psi_{XO}) \quad \text{-----(21)}$$

$$\text{Along } C = -dx/dt; (M_{XP} - M_{XR}) + (M_{YXP} - M_{YXR}) = \Delta x/2(Q_{XP} + Q_{XL}) - \Delta y/2 \rho I C (\psi_{XP} - 2\psi_{XR} + \psi_{XO}) \quad \text{-----(22)}$$

Flexural waves – Angular y- direction:

$$\text{Along } C = dy/dt; (M_{YP} - M_{YL}) - (M_{XYP} - M_{XYL}) = -\Delta y/2(Q_{YP} + Q_{YL}) + \Delta x/2 \rho I C (\psi_{YP} - 2\psi_{YL} + \psi_{YO}) \quad \text{-----(23)}$$

$$\text{Along } C = -dy/dt; (M_{YP} - M_{YR}) + (M_{XYP} - M_{XYR}) = \Delta y/2(Q_{YP} + Q_{YL}) - \Delta x/2 \rho I C (\psi_{YP} - 2\psi_{YR} + \psi_{YO}) \quad \text{-----(24)}$$

Where:

$\Delta x, \Delta y$ = the finite difference intervals in x and y directions respectively.

F_X, F_Y = in-plane forces in x and y directions respectively at node (i) and time (t1) in plates.

F_{XP}, F_{YP} = in-plane forces in x and y directions respectively at node (i) and time (t2).

F_{XL}, F_{YL} = in-plane forces in x and y directions respectively at node (i-1) and time (t1).

F_{XR}, F_{YR} = in-plane forces in x and y directions respectively at node (i+1) and time (t1).

Q_X, Q_Y = shear forces in xz and yz planes respectively at node (i) and time (t1).

Q_{XP}, Q_{YP} = shear forces in xz and yz planes respectively at node (i) and time (t2).

Q_{XL}, Q_{YL} = shear forces in xz and yz planes respectively at node (i-1) and time (t1).

Q_{XR}, Q_{YR} = shear forces in xz and yz planes respectively at node (i+1) and time (t1).

M_X, M_Y = bending moments about x and y axis respectively at node (i) and time (t1) in plates.

M_{XP}, M_{YP} = bending moments about x and y axis respectively at node (i) and time (t2).

M_{XL}, M_{YL} = bending moments about x and y axis respectively at node (i-1) and time (t1).

M_{XR}, M_{YR} = bending moments about x and y axis respectively at node (i+1) and time (t1).

M_{XY}, M_{YX} = twisting moments at node (i) and time (t1) in plates.

M_{XYP}, M_{YXP} = twisting moments at node (i) and time (t2).

M_{XYL}, M_{YXL} = twisting moments at node (i-1) and time (t1).

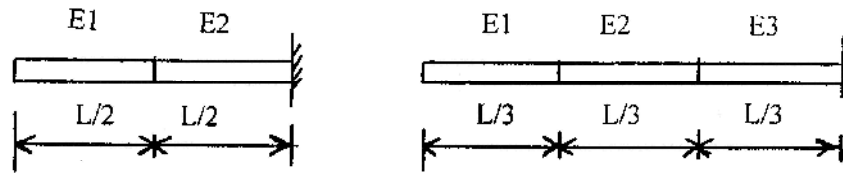
M_{XYR}, M_{YXR} = twisting moments at node (i+1) and time (t1).

Results And Discussions:

In this study, the (MOC) is applied on beams with two or more material and the method is extended to be applicable on two dimensional problems such as plated. A cantilever beam, subjected to a suddenly applied axial load, the load is applied to an initially unloaded member at time zero for beams with one, two and three different material as shown in Fig.(2). And the beam cross section area $A=25 \text{ mm}^2$, Moment of inertia $I=10000 \text{ mm}^4$, length $L=1\text{m}$, material density $\rho=8000 \text{ kg/m}^3$.

The following variations of materials are considered:

- 1- $E1 = E2 = 200 \text{ GPa}$ (steel).
- 2- $E1 = E2 = 70 \text{ GPa}$ (aluminum).
- 3- $E1 = 70 \text{ GPa}$ and $E2 = 200 \text{ GPa}$. The beam divided into two equal parts.
- 4- $E1 = 200 \text{ GPa}$ and $E2 = 70 \text{ GPa}$. The beam divided into two equal parts.
- 5- $E1 = E3 = 200 \text{ GPa}$ and $E2=70 \text{ GPa}$. The beam divided into three equal parts.
- 6- $E1 = E3 = 70 \text{ GPa}$ and $E2=200 \text{ GPa}$. The beam divided into three equal parts.



(a): Beam divided into two equal parts. (b): Beam divided into three equal parts.

Fig.(2): Multi – material beams.

The resulting axial force at the fixed end and displacement at the tip are shown in figs. (3 and 4) for cases (1-4), while responses of cases (5 and 6) are shown in figs. (5 and 6). The results show that the accuracy of solution is achieved with (4) number of grids in axial waves and at least (32) grids in flexural waves, so the number of grids used in the analysis is (4) in axial waves and (32) grids in flexural waves. The axial waves propagate in unchanged form in one material beam, while it changed it's form in multi-material beams. Also the figure shows that case (5) gives the same natural frequency as in case (1), the reduction in the frequency is (4%) in case (3) and become (40%) in case (2 and 6), while in case (4) the reduction become (52.3%). The natural frequency (ω_n) and time period (T_n) for all cases are shown in following table for cantilever beam subjected to axial load:

Table(1) axial natural frequency (ω_n) and time period (T_n) for cantilever beam.

Case	Description	ω_n rad/sec	T_n msec
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1	E1 = E2 = 200 GPa.	7854	0.8
2	E1 = E2 = 70 GPa.	4646.8	1.35
3	E1 = 70 GPa and E2 = 200 GPa	7540	0.833
4	E1 = 200 GPa and E2 = 70 GPa	3744.5	1.628
5	E1 = E3 = 200 GPa and E2=70 GPa	7854	0.8
6	E1 = E3 = 70 GPa and E2=200 GPa	4692.8	1.34

The natural frequency of case 1 and 2 are exactly equal to that predicted by analytical method given in [12] ($\omega_n = \pi/2L \sqrt{E/\rho}$). In case (5) the material of the beam (E=200 GPa) is changed to (E=70 GPa) on the beam length (L/3), the natural frequency of the beam remains constant in comparison to case (1), also In case (6) the material of the beam (E=70 GPa) is changed to (E=200 GPa) on the beam length (L/3), the natural frequency of the beam close enough to case (2).

figs.(7, 8 and 9) show the dynamic response of the cantilever beam subjected to shear force at the tip, using coupled axial and flexural characteristics equations. The results show that the flexural waves propagate in changing form in both one and multi-material members and the t-wise interpolation method is necessary in the analysis to achieve the required accuracy. Also showed that reduction in the natural frequency is (17%) in case (3) and (38%) in case (4), while the reduction becomes (41%) in case(2). The natural frequency (ω_n) and time period (T_n) for all cases are shown in following table for cantilever beam subjected to transverse shear load:

Table(2) flexural natural frequency (ω_n) and time period (T_n) for cantilever beam.

Case	Description	ω_n rad/sec	T_n msec
1	E1 = E2 = 200 GPa.	628.9	10
2	E1 = E2 = 70 GPa.	372	16.9
3	E1 = 70 GPa and E2 = 200 GPa	523.6	12
4	E1 = 200 GPa and E2 = 70 GPa	392.7	16

The natural frequency of case 1 and 2 are exactly equal to that predicted by analytical method given in [12] $\{\omega_n = (1.875/L)^2 \sqrt{EI/\rho A}\}$.

The effect of beams with different materials on the propagation of flexural waves is more than axial waves because the behavior and nature of flexural waves are more complicated than axial waves. And the flexural waves consist of two types of waves (bending and shear waves) which propagate in the material with different speeds.

Figs. (10 and 11) show the dynamic response of fixed end reaction and the deflection at the free end of a cantilever plate subjected to a suddenly constant in-plane force in one direction, the plate is divided to 16 elements. The resulting response is exactly same as a cantilever beam.

Figs. (12 and 13) show the dynamic response of the in-plane force and deflection at the center of the plate fixed at two adjacent edges and free at the opposite two edges and subjected to a suddenly constant in-plane force in two directions.

Conclusions:

1-The accuracy of solution is achieved with (4) number of grids in axial waves, while it needs to at least (32) grids in flexural waves.

2-The axial waves propagate in unchanged form in one material beam, while it changed its form in multi-material beams.

3-The flexural waves propagate in a changing form in both one and multi-material members.

4-The t-wise interpolation method is necessary in the analysis to achieve the required accuracy.

5-In axial waves and single material beams, the graphical response of vibration shows the fundamental natural frequency only, while in flexural waves and multi-material members, the graphical response shows the higher frequencies in addition to the fundamental frequency.

6-The effect of multi-material on flexural waves is more than axial waves because the behavior and nature of flexural waves are more complicated than axial waves. And the flexural waves consist of two types of waves (bending and shear waves) which propagate in the material with different speeds.

7-In axial wave, the behavior of the beam is not changed when the material of the beam is changed over a part of the beam equal to $(L/3)$.

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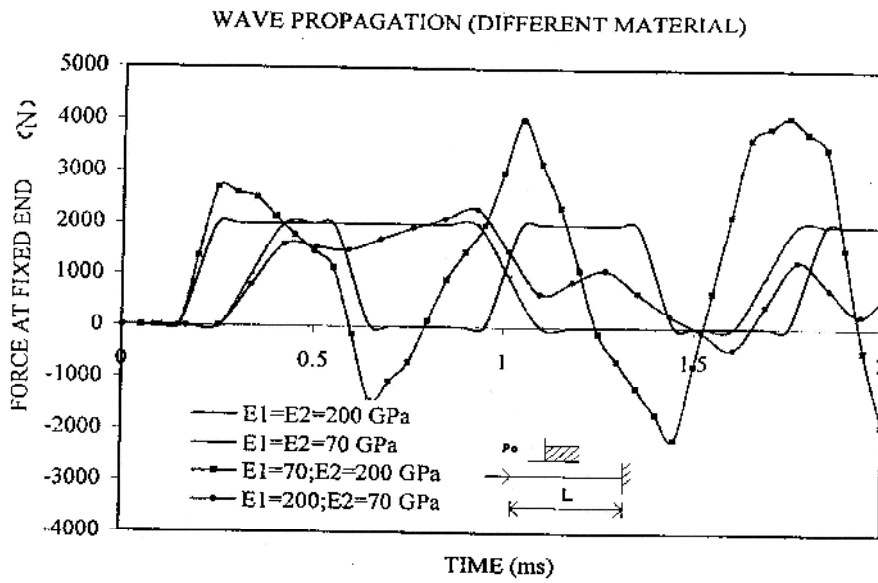


Fig.(3): Dynamic response of the force at fixed end.

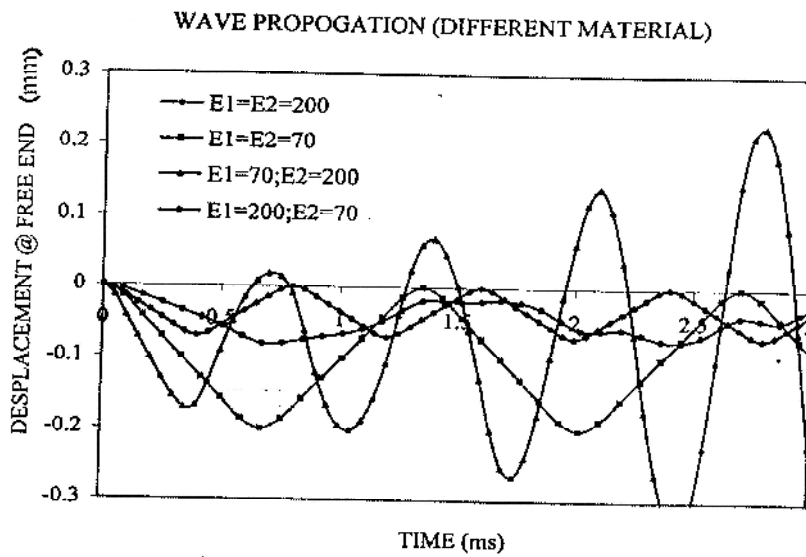


Fig.(4): Dynamic response of the displacement at free end.

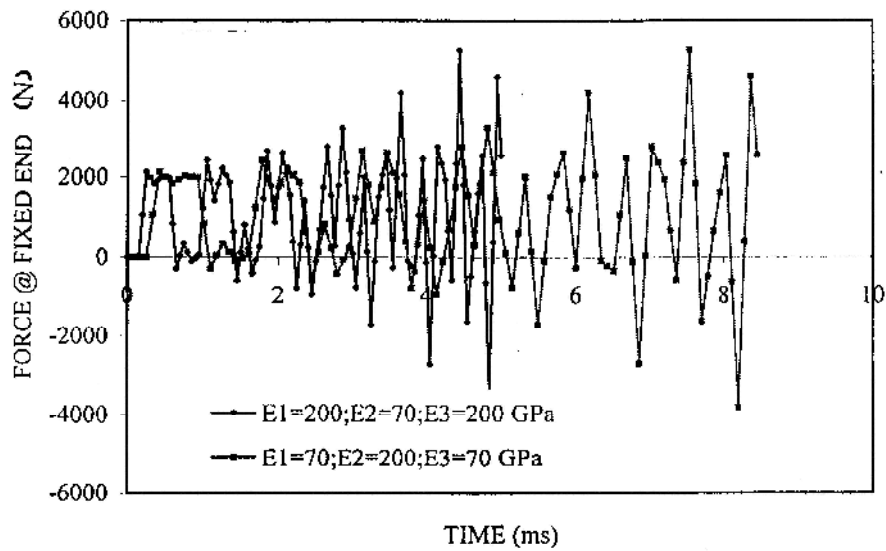


Fig.(5): Dynamic response of the force at fixed end.

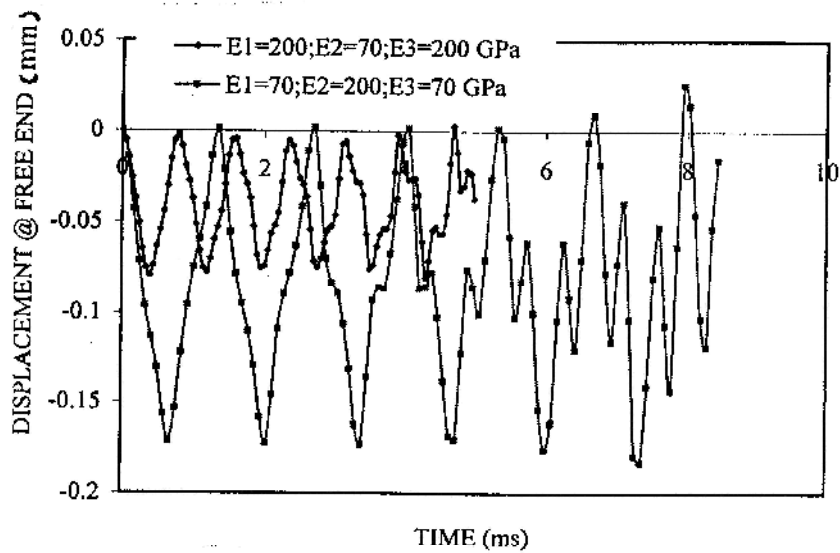


Fig.(6): Dynamic response of the displacement at free end.

$A=125\text{mm}^2$, $I=10000\text{mm}^4$, $\rho=8000\text{kg/m}^3$, $L=1\text{m}$, $P_0=1000\text{N}$

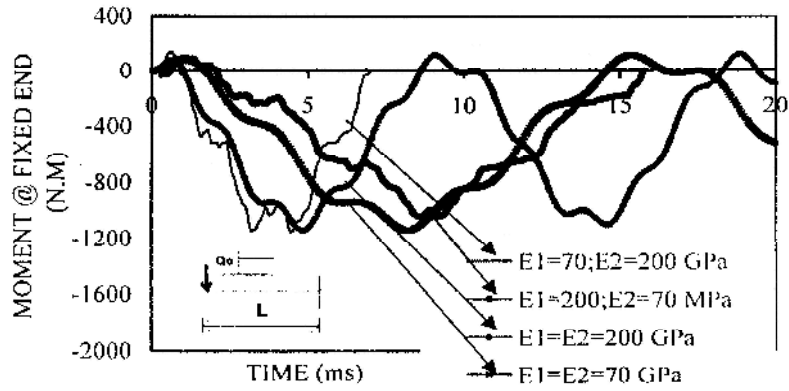


Fig.(7): Dynamic response of the moment at fixed end.

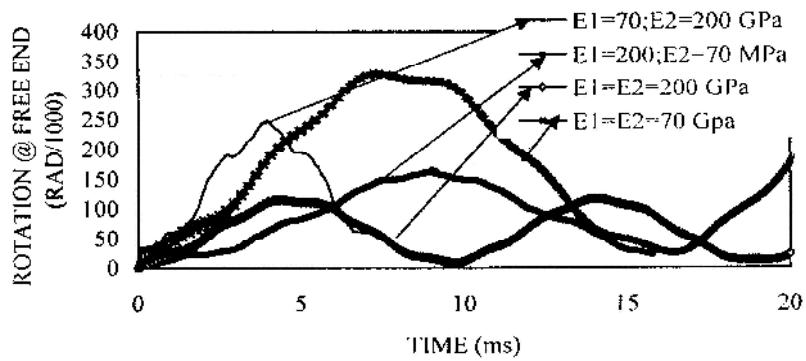


Fig.(8): Dynamic response of the rotation at free end.

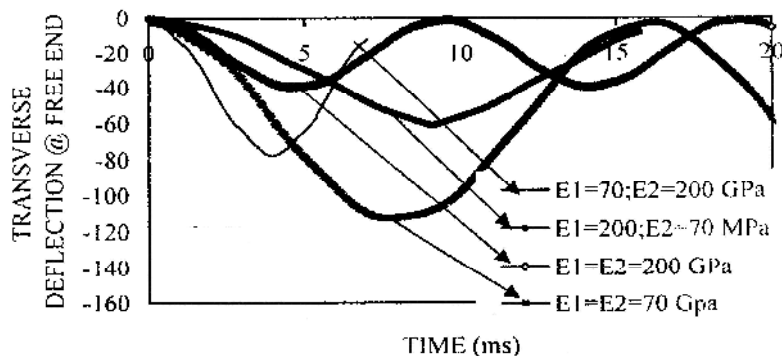


Fig.(9): Dynamic response of the transverse deflection at free end.

$\Lambda=125\text{mm}^2$, $I=10000\text{ mm}^4$, $\rho=8000\text{ kg/m}^3$, $L=0.5\text{ m}$, $Q_0=1000\text{ N}$

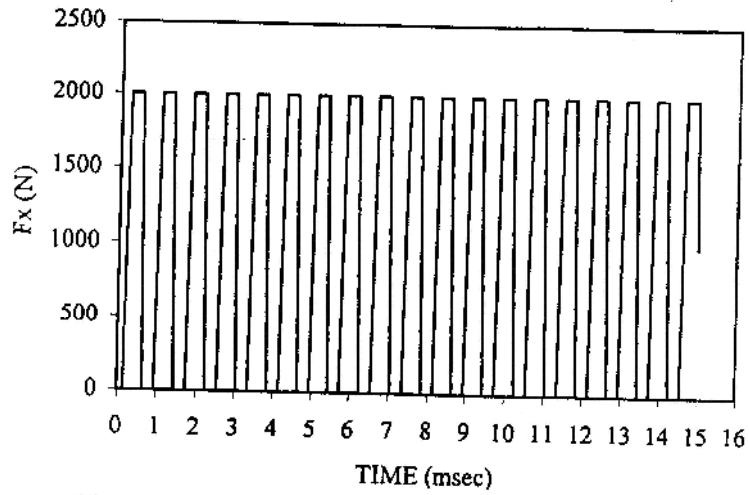
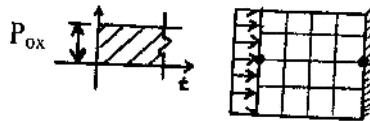


Fig.(10): Dynamic response of the in-plane force at fixed end.



$E=200 \text{ GPa}$, $\rho=8000 \text{ kg/m}^3$, $h=0.1 \text{ m}$, $\nu=0.25$, $L_x=L_y=1\text{m}$, $P_{ox}=1000\text{N}$

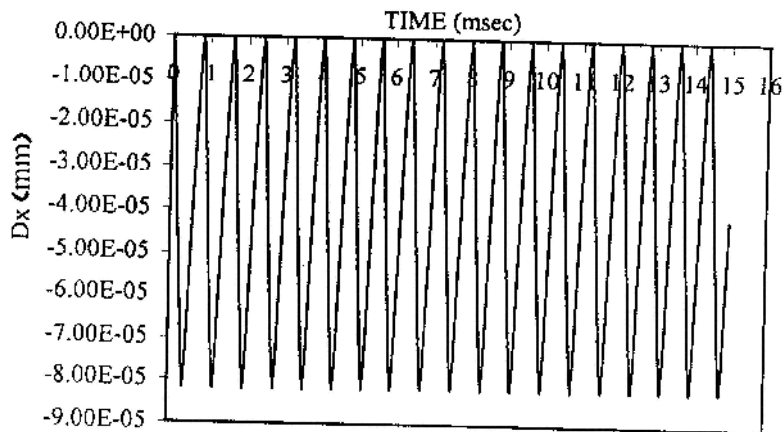


Fig.(11): Dynamic response of the in-plane displacement at free end.

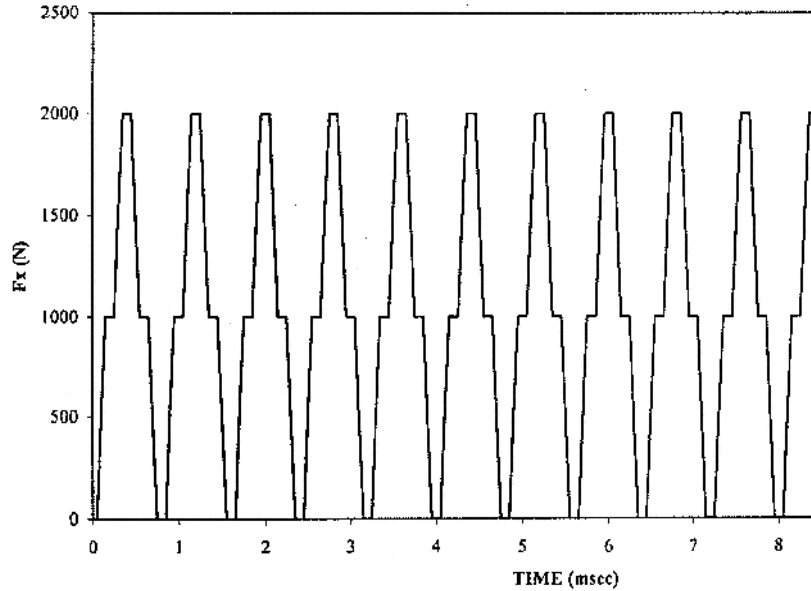


Fig.(12): Dynamic response of the in-plane force at the center.

$E=200 \text{ GPa}$, $\rho=8000 \text{ kg/m}^3$, $h=0.1 \text{ m}$,
 $\nu=0.25$, $L_x=L_y=1 \text{ m}$, $P_{ox}=P_{oy}=1000 \text{ N}$

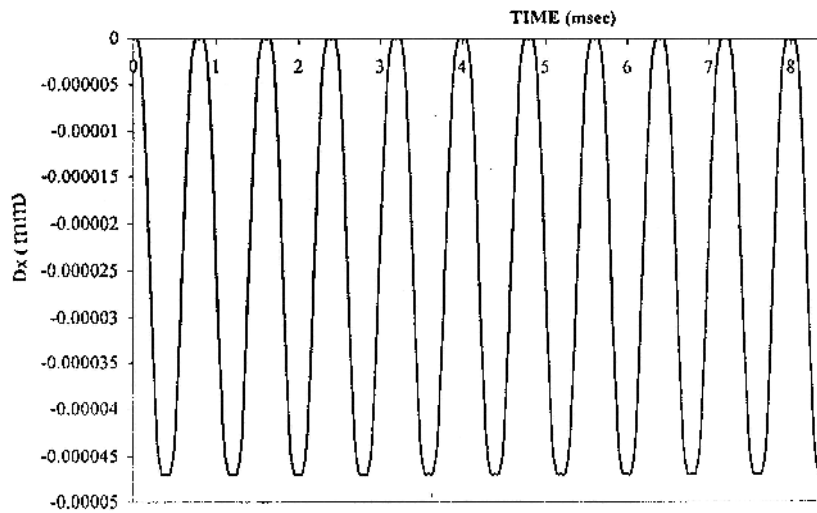
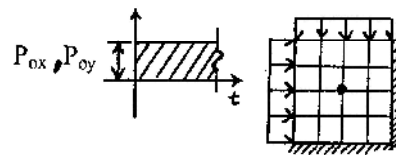


Fig.(13): Dynamic response of the in-plane displacement at the center.

